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A Note on a Problem of Rivlin

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1. INTRODUCTION

The following problem was posed by Rivlin [1]. Characterize those n-tuples of algebraic polynomials $\{p_0, p_1, \dots, p_{n-1}\}$ such that the degree of p_j is $j\,(j = 0, 1, 2, \dots, n-1)$ for which there exists an $x \in C([0, 1])$ (the space of continuous real-valued functions on [0, 1]) so that the polynomial of best approximation of degree j to x in the sense of Chebyshev is p_j. The purpose of this note is to prove the following theorem:

THEOREM. *Let $E = C([0, 1])$ and let $\{f_1, f_2\}$ be any Chebyshev system. Let $g_1 \in G_1$, the span of $\{f_1\}$, and $g_2 \in G_2$, the span of $\{f_1, f_2\}$. In order that there exists an $x \in C([0, 1])$ such that*

$$\pi_{G_k}(x) = g_k \qquad (k = 1, 2),$$

it is necessary and sufficient that either $g = g_2 - g_1$ is identically equal to zero or that g changes sign once in [0, 1], *where π_{G_k} is the metric projection on to the subspace G_k.*

Remark. This theorem generalizes theorem 7 of [2] where the result was proved with $G_1 = \text{span}\,\{1\}$ and $G_2 = \text{span}\,\{1, t\}$.

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2. PROOF OF THE THEOREM

The proof of the necessity of the condition proceeds as in [2]. We shall prove only the sufficiency part. Suppose g changes sign once in [0, 1]. Choose

$\alpha \in (0, 1)$ such that $g(\alpha) = 0$. We can choose, then, two points θ_1 and θ_2, $0 < \theta_1 < \theta_2 < 1$, such that $g(\theta_1) = -g(\theta_2)$ and such that $g(t)$ lies between $g(\theta_1)$ and $g(\theta_2)$ when $t \in [\theta_1, \theta_2]$, i.e., either $g(\theta_1) \leqslant g(t) \leqslant g(\theta_2)$ or

$$g(\theta_1) \geqslant g(t) \geqslant g(\theta_2).$$

We assume for the moment the first possibility. Then $g(\theta_1) < 0$. Let M be a real number $\geqslant 2\|g\|$. We define a continuous function y_2 on $[0, 1]$ as follows. Let $\theta_2 < t_0 < 1$. Set

$$y_2(t) = \begin{cases} (M - 2\,|\,g(t) - g(\theta_1)|)(t/\theta_1), & \text{for} \quad 0 \leqslant t \leqslant \theta_1\,; \\ [M/(\theta_2 - \theta_1)](\theta_1 + \theta_2 - 2t), & \text{for} \quad \theta_1 \leqslant t \leqslant \theta_2\,; \\ (-M + 2\,|\,g(t) - g(\theta_2)|)[(t_0 - t)/(t_0 - \theta_2)], & \text{for} \quad \theta_2 \leqslant t \leqslant t_0\,; \\ [M/(1 - t_0)](t - t_0), & \text{for} \quad t_0 \leqslant t \leqslant 1. \end{cases}$$

Then $y_2(0) = 0 = y_2(t_0)$, $y_2(\theta_1) = M = -y_2(\theta_2) = y_2(1)$ and y_2 is continuous on $[0, 1]$. Put $y_1(t) = y_2(t) - g(t)$ for $0 \leqslant t \leqslant 1$.

We claim that y_1 and y_2, so constructed, satisfy the hypothesis of Theorem 1 of [2] and then the proof follows by an application of that theorem. It is easy to verify that $|\,y_2(t)| \leqslant M$ for all $t \in [0, 1]$. As to y_1, we have the following:

(i) Let $0 \leqslant t \leqslant \theta_1$; since g is negative,

$$\begin{aligned} |\,y_1(t)| = y_1(t) &= (M - 2\,|\,g(\theta_1) - g(t)|)(t/\theta_1) - g(t). \\ &\leqslant M - 2\,|\,g(\theta_1) - g(t)| - g(t). \\ &= M - g(\theta_1) - 2\,|\,g(\theta_1) - g(t)| + g(\theta_1) - g(t) \\ &\leqslant M - g(\theta_1) = y_1(\theta_1). \end{aligned}$$

(ii) Let $\theta_1 \leqslant t \leqslant \theta_2$; then $g(\theta_1) \leqslant g(t) \leqslant g(\theta_2)$, and

$$\begin{aligned} |\,y_1(t)| &\leqslant [M/(\theta_2 - \theta_1)](\theta_1 + \theta_2 - 2t) + |\,g(t)| \\ &\leqslant M + |\,g(t)| \leqslant M + g(\theta_2) = M - g(\theta_1) = y_1(\theta_1). \end{aligned}$$

(iii) Let $\theta_2 \leqslant t \leqslant t_0$; then $g(t) \geqslant 0$, and

$$\begin{aligned} |\,y_1(t)| &\leqslant M - 2\,|\,g(\theta_2) - g(t)| + g(t) \\ &= M + g(\theta_2) - 2\,|\,g(\theta_2) - g(t)| + g(t) - g(\theta_2) \\ &\leqslant M + g(\theta_2) = M - g(\theta_1) = y_1(\theta_1). \end{aligned}$$

(iv) Let $t_0 \leqslant t \leqslant 1$; then

$$\begin{aligned} |\,y_1(t)| &= |[M/(1 - t_0)](t - t_0) - g(t)| \\ &\leqslant M < y_1(\theta_1). \end{aligned}$$

Thus we have $y_1(\theta_1) = -y_1(\theta_2) = \| y_1 \|$. We have already noticed that $y_2(\theta_1) = M = -y_2(\theta_2) = y_2(1) = \| y_2 \|$. From a well-known theorem of Chebyshev it then follows that $y_1 \in \pi_{G_1}^{-1}(0)$ and $y_2 \in \pi_{G_2}^{-1}(0)$, where

$$\pi_G^{-1}(0) = \{ y \in C([0, 1]) : 0 \in \pi_G(y)\};$$

that is, y_1 and y_2 satisfy the condition of Theorem 1 of [2]. A slight modification of the above construction gives the proof for the case $g(\theta_1) \geqslant g(t) \geqslant g(\theta_2)$. This completes the proof.

References

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2. Frank Deutsch, Peter D. Morris, and Ivan Singer, On a problem of T. J. Rivlin in approximation theory, *J. Approximation Theory* **2** (1969), 342–354.